The alternating group

We can now use our new tools to understand an important subgroup of S_n called the alternating group, or A_n.

First we need to define the "sign" of an element in S_n. Recall that we can think of an element of S_n as a permutation of $1\ 2\ 3 \dots n$.

i.e. $\sigma \in S_n$ corresponds to the permutation $\sigma(1)\sigma(2)\dots\sigma(n)$.

Intuitively, we can see that we can get from one permutation to another by successively switching two elements.

Ex: To get from $1\,2\,3\,4\,5$ to $4\,2\,1\,3\,5$:

$$1\,2\,3\,4\,5 \xrightarrow[\underset{\sigma_1}{\shortparallel}]{(13)} 3\,2\,1\,4\,5 \xrightarrow[\underset{\sigma_2}{\shortparallel}]{(34)} 4\,2\,1\,3\,5 \qquad (34)(13) = (1\,4\,3)$$

$$\sigma_1(3) = 1,\ \sigma_2(1) = 1 \Rightarrow \sigma_2(\sigma_1(3)) = 1.\ \checkmark$$

To get from $1\,2\,3\,4\,5$ to $3\,2\,4\,5\,1$: $(1\,3\,4\,5)$

$$1\,2\,3\,4\,5 \xrightarrow[(13)]{} 3\,2\,1\,4\,5 \xrightarrow[(14)]{} 3\,2\,4\,1\,5 \xrightarrow[(15)]{} 3\,2\,4\,5\,1 \qquad (15)(14)(13) = (1\,3\,4\,5)\ \checkmark$$

Def: A 2-cycle is called a transposition.

We can write every m-cycle as a product of 2-cycles as follows:

$$(a_1\, a_2 \dots a_m) = (a_1 a_m)(a_1 a_{m-1}) \dots (a_1 a_2)$$

Since every element is the product of cycles, every element of S_n can be written as the product of transpositions.

Note that there is not a unique way to do this!

e.g. $(1\,2)(1\,3) = (1\,3\,2) = (2\,3)(1\,2)$ and $1 = (1\,2)(1\,2) = (1\,2)^2(2\,3)^2$.

However, we can determine the parity (i.e. odd or even) of the # of terms, and this will be fixed for a given permutation:

Def: If $\sigma \in S_n$, the sign of σ is 1 if σ can be written as the product of an even # of transpositions, and -1 if σ is the product of an odd #.

Claim: The sign of $\sigma \in S_n$ is well-defined. That is, any decomposition of σ into transpositions has the same parity (odd or even).

We won't prove this, but see D+F or any other algebra book for a proof.

Prop: The map $\varepsilon : S_n \to \{\pm 1\}$ where each element is sent to its sign is a homomorphism ($\{\pm 1\}$ is a group under multiplication).

Pf: If $\sigma, \tau \in S_n$ and they can be written as the product of m and n transpositions, respectively, then $\sigma\tau$ can be written as $m+n$ transpositions.

If one of m and n is odd, the other even, then $m+n$ is odd,

so $\varepsilon(\sigma\tau) = -1 = 1 \cdot -1 = \varepsilon(\sigma)\varepsilon(\tau)$

Otherwise, $m+n$ is even and $\varepsilon(\sigma\tau) = 1 = (\pm 1)^2 = \varepsilon(\sigma)\varepsilon(\tau)$. $\square$

Def: The kernel of ε is called the alternating group of degree n, denoted A_n. Thus, $A_n \trianglelefteq S_n$ and A_n consists of all the even permutations.

Note that a cycle $(a_1 \cdots a_m) = (a_1\, a_m)(a_1\, a_{m-1})\cdots(a_1\, a_2)$ is even iff m is odd.

Ex: $A_1 = 1$, $A_2 = 1$, $A_3 = \{1, (1\,2\,3), (1\,3\,2)\} \cong \mathbb{Z}_3$

A_4 has 12 elements, the even elements of S_4:

$1, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3), (1\,2\,3), (1\,2\,4), (1\,3\,2), (1\,3\,4), (1\,4\,2), (1\,4\,3), (2\,3\,4), (2\,4\,3)$, 3 elements of order 2, 8 of order 3, so it's not isomorphic to D_{12}.

In fact, it's isomorphic to the group of symmetries of a tetrahedron!